Values of the constants used in the present case, are:

$\beta_0 = -{}^5/_{12},\quad \beta_1 = {}^{11}/_{180},\quad \beta_2 = -{}^{239}/_{12096},\quad D = 2.549,\quad A_n = B_n\ (n = 1,\ldots, m).$

The error of the approximation (2.7) does not exceed 3% for all $0 \leqslant \operatorname{Re} s < \infty$.

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A GENERALIZATION OF FOURIER'S INTEGRAL THEOREM AND ITS APPLICATIONS

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I. T. EFIMOVA and Ia. S. UFLIAND
(Leningrad)

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1. Solution of certain classes of the boundary value problems of mathematical physics for a two-layer medium demands that the given function be expanded into an integral in terms of the functions

$$\varphi(x,\lambda) = \begin{cases} \mu \sin\sqrt{\beta_1}\,\lambda x & (0 < x < l) \\ \sin\sqrt{\beta_1}\lambda l \cos\sqrt{\beta_2}\lambda(x-l) + \delta\cos\sqrt{\beta_1}\,\lambda l \sin\sqrt{\beta_2}\lambda(x-l) & (l < x < \infty) \end{cases} \tag{1.1}$$

which are eigenfunctions of the following singular boundary value problem:

$$\varphi'' + \beta_1\lambda^2\varphi = 0 \quad (0 < x < l), \quad \varphi'' + \beta_2\lambda^2\varphi = 0 \quad (l < x < \infty) \tag{1.2}$$

$$\varphi(0) = 0, \quad \varphi(\infty) < \infty, \quad \varphi(l-0) = \mu\varphi(l+0), \quad \varphi'(l-0) = \nu\varphi'(l+0)$$

The fundamental result of the present investigation can be stated in the form of the following theorem: if $f(x)$ is a piece-wise continuous function absolutely integrable on the interval $(0, \infty)$ and possessing a bounded variation in this interval, then

$$\frac{2}{\pi}\int_0^\infty \frac{\varphi(x,\lambda)}{\omega(\lambda)}\,d\lambda \int_0^\infty f(\xi)\,r(\xi)\,\varphi(\xi,\lambda)\,d\xi = \begin{cases} {}^1/_2\,[f(x-0) + f(x+0)] & (x \neq l) \\ [\delta f(l-0) + \mu f(l+0)]\,/\,1+\delta & (x = l) \end{cases} \tag{1.3}$$

$$\omega(\lambda) = \sin^2 \sqrt{\beta_1}\lambda l + \delta^2 \cos^2 \sqrt{\beta_1}\lambda l, \quad \delta = (\mu/\nu)\sqrt{\beta_1/\beta_2} \tag{1.4}$$

$$r(x) = \delta \sqrt{\beta_1}/\mu^2 \quad (0 < x < l), \quad r(x) = \sqrt{\beta_2} \quad (l < x < \infty) \tag{1.5}$$

When $\beta_1 = \beta_2 = \mu = \nu = 1$, the expansion (1.3) becomes an ordinary Fourier integral.

To prove the theorem, we shall consider the integral

$$J(x, T) = \frac{2}{\pi}\int_0^T d\lambda \int_0^\infty f(\xi) r(\xi) \frac{\varphi(\xi, \lambda)\varphi(x, \lambda)}{\omega(\lambda)} d\xi \tag{1.6}$$

after noting, that

$$\left| f(\xi) r(\xi) \frac{\varphi(\xi, \lambda)\varphi(x, \lambda)}{\omega(\lambda)} \right| \leqslant C|f(\xi)| \qquad x, \xi \in (0, \infty)$$

By virtue of the above inequality and of the absolute integrability of $f(x)$, the inner integral of (1.6) converges uniformly in λ for all $\lambda \in (0, \infty)$.

Therefore,

$$J(x, T) = \frac{2}{\pi}\int_0^\infty f(\xi) r(\xi) d\xi \int_0^T \frac{\varphi(\xi, \lambda)\varphi(x, \lambda)}{\omega(\lambda)} d\lambda =$$

$$= \frac{2}{\pi}\frac{\delta\sqrt{\beta_1}}{\mu^2}\int_0^l f(\xi) d\xi \int_0^T \frac{\mu \sin \sqrt{\beta_1}\lambda\xi\varphi(x,\lambda)}{\omega(\lambda)} d\lambda +$$

$$+ \frac{2}{\pi}\sqrt{\beta_2}\int_l^\infty f(\xi) d\xi \int_0^T [\sin \sqrt{\beta_1}\lambda \cos \sqrt{\beta_2}\lambda(\xi - l) + \delta \cos \sqrt{\beta_1}\lambda l \sin \sqrt{\beta_2}\lambda(\xi - l)] \times$$

$$\times \frac{\varphi(x, \lambda)}{\omega(\lambda)} d\lambda = \frac{2}{\pi}\delta\sqrt{\beta_1}\int_0^l f(\xi)\psi_1(x, \xi, T) d\xi + \frac{2}{\pi}\sqrt{\beta_2}\mu\int_l^\infty f(\xi)\psi_2(x, \xi, T) d\xi \tag{1.7}$$

To start with, let $x \in (0, l)$. Then

$$\psi_1 = \int_0^T \frac{\sin \sqrt{\beta_1}\lambda x \sin \sqrt{\beta_1}\lambda\xi}{\omega(\lambda)} d\lambda \tag{1.8}$$

$$\psi_2 = \int_0^T \sin \sqrt{\beta_1}\lambda x [\sin \sqrt{\beta_1}\lambda l \cos \sqrt{\beta_2}\lambda(\xi - l) + \delta \cos \sqrt{\beta_1}\lambda l \sin \sqrt{\beta_2}\lambda(\xi - l)] \frac{d\lambda}{\omega(\lambda)} \tag{1.9}$$

Expanding now the quantity $1/\omega(\lambda)$ into a series in powers of the parameter $\varepsilon \cos 2\sqrt{\beta_1}\lambda l$ where

$$\varepsilon = (1 - \delta^2)/(1 + \delta^2) \tag{1.10}$$

changing the order of summation and integration in (1.8) and representing the powers of $\cos 2\sqrt{\beta_1}\lambda l$ by multiple arcs, we can complete the quadratures in λ and reduce (1.8) after some manipulations, to the form

$$(1 + \delta^2)\psi_1 = \left[\frac{\sin \theta_- T}{\theta_-} - \frac{\sin \theta_+ T}{\theta_+}\right]\frac{1 + \delta^2}{2\delta} + \sum_{n=1}^\infty \frac{\varepsilon^n}{2^n}\sum_{k=0}^{\nu_n} C_n^k \left[\frac{\sin(\theta_- - \alpha_n)T}{\theta_- - \alpha_n} - \right.$$

$$\left. - \frac{\sin(\theta_+ - \alpha_n)T}{\theta_+ - \alpha_n} + \frac{\sin(\theta_- + \alpha_n)T}{\theta_- + \alpha_n} - \frac{\sin(\theta_+ + \alpha_n)T}{\theta_+ + \alpha_n}\right] \tag{1.11}$$

where

$$\theta_\pm = \sqrt{\beta_1}(x \pm \xi), \quad \alpha_n = 2l\sqrt{\beta_1}(n - k), \quad \nu_n = \text{ent}\,[{}^1/_2\,(n - 1)]$$

Similarly, setting $\eta = \sqrt{\beta_2}(\xi - l)$, $\vartheta_\pm = \sqrt{\beta_1}\,(l \pm x)$, we obtain from (1.9)

$$2\psi_2 = \frac{1 + \delta}{2}\left[\frac{\sin(\vartheta_- + \eta)T}{\vartheta_- + \eta} - \frac{\sin(\vartheta_+ + \eta)T}{\vartheta_+ + \eta}\right] + \frac{1 - \delta}{2}\left[\frac{\sin(\vartheta_- - \eta)T}{\vartheta_- - \eta} - \right.$$

$$-\frac{\sin(\vartheta_+ - \eta)T}{\vartheta_+ - \eta}\Big] + \frac{1+\delta}{1+\delta^2}\sum_{n=1}^{\infty}\frac{\varepsilon^n}{2^n}\sum_{k=0}^{\nu_n} C_n{}^k \Big[\frac{\sin(\vartheta_- + \eta - \alpha_n)T}{\vartheta_- + \eta - \alpha_n} -$$

$$-\frac{\sin(\vartheta_+ + \eta - \alpha_n)T}{\vartheta_+ + \eta - \alpha_n} + \frac{\sin(\vartheta_- + \eta + \alpha_n)T}{\vartheta_- + \eta + \alpha_n} - \frac{\sin(\vartheta_+ + \eta + \alpha_n T)}{\vartheta_+ + \eta + \alpha_n}\Big] +$$

$$+\frac{1-\delta}{1+\delta^2}\sum_{n=1}^{\infty}\frac{\varepsilon^n}{2^n}\sum_{k=0}^{\nu_n} C_n{}^k \Big[\frac{\sin(\vartheta_- - \eta - \alpha_n)T}{\vartheta_- - \eta - \alpha_n} - \frac{\sin(\vartheta_+ - \eta - \alpha_n)T}{\vartheta_+ - \eta - \alpha_n} +$$

$$+\frac{\sin(\vartheta_- - \eta + \alpha_n)T}{\vartheta_- - \eta + \alpha_n} - \frac{\sin(\vartheta_+ - \eta + \alpha_n)T}{\vartheta_+ - \eta + \alpha_n}\Big] \tag{1.12}$$

Passing now to the limit as $T \to \infty$ in (1.7), (1.11) and (1.12) and applying the Riemann and Dirichlet lemmas [1], we obtain after certain transformations,

$$\lim_{T\to\infty} J(x, T) = {}^1/_2 [f(x-0) + f(x+0)] + \frac{\sqrt{\beta_2}}{\pi(1+\delta^2)}\lim_{T\to\infty}\int_l^{\infty} f(\xi)\sum_{m=1}^{\infty}\frac{\varepsilon^{2m}}{2^{2m}}\sum_{k=0}^{m-1} C_{2m}^k \times$$

$$\times\Big\{\Big[1+\delta-(1-\delta)\frac{2m-k}{\varepsilon m}\Big]\Big[\frac{\sin\{\sqrt{\beta_1}[(4m-4k-1)l+x]-\eta\}T}{\sqrt{\beta_1}[(4m-4k-1)l+x]-\eta} -$$

$$-\frac{\sin\{\sqrt{\beta_1}[(4m-4k-1)l-x]-\eta\}T}{\sqrt{\beta_1}[(4m-4k-1)l-x]-\eta} - \Big[1-\delta-(1+\delta)\frac{k}{\varepsilon m}\Big]\times \tag{1.13}$$

$$\times\frac{\sin\{\sqrt{\beta_1}[(4m-4k+1)l+x]-\eta\}T}{\sqrt{\beta_1}[(4m-4k+1)l+x]-\eta} - \frac{\sin\{\sqrt{\beta_1}[(4m-4k+1)l-x]-\eta\}T}{\sqrt{\beta_1}[(4m-4k+1)l-x]-\eta}\Big\}d\xi$$

Changing the order of summation over m and k in the last expression yields an inner sum of the form

$$\sum_{m=k}^{\infty}\frac{\varepsilon^{2m}}{2^{2m}} C_{2m}^{m-k}\Big[\delta(\varepsilon+1)-\frac{k}{m}\Big] \tag{1.14}$$

which, by virtue of the equations

$$\Big(\frac{1-\sqrt{1-x^2}}{x}\Big)^{2k} = \sqrt{1-x^2}\sum_{m=k}^{\infty} C_{2m}^{m-k}\Big(\frac{x}{2}\Big)^{2m} = k\sum_{m=k}^{\infty}\frac{C_{2m}^{m-k}}{m}\Big(\frac{x}{2}\Big)^{2m} \tag{1.15}$$

is equal to zero. Thus, we have

$$\lim J(x, T) = {}^1/_2 [f(x-0) + f(x+0)], \quad T \to \infty$$

Q.E.D. Proof of the theorem for the cases $x \in (l, \infty)$ and $x = l$, is analogous.

2. Examples of application of the above expansion to certain steady state problems of mathematical physics and to the theory of elasticity for a two-layer medium, follow.

1°. We require to solve the Laplace's equation for the function $u(r, z)$ inside a semi-infinite cylinder of radius R with the conditions:

$$u(r, 0) = 0,\ u(r, l-0) = \mu u(r, l+0),\ u_z(r, l-0) = \nu u_z(r, l+0,\ u(R, z) = f(z) \tag{2.1}$$

Separation of variables yields

$$u(r, z) = \int_0^{\infty} A(\lambda)\frac{I_0(\lambda r)}{I_0(\lambda R)}\varphi(z, \lambda)\, d\lambda \tag{2.2}$$

and the inhomogeneous boundary condition leads to the expansion

$$f(z) = \int_0^{\infty} A(\lambda)\varphi(z, \lambda)\, d\lambda \tag{2.3}$$

By (1.3) we have $(\beta_1 = \beta_2 = 1)$

$$A(\lambda)=\frac{2}{\pi}\frac{1}{\omega(\lambda)}\int_0^\infty f(\xi)\,r(\xi)\,\varphi(\xi,\lambda)\,d\xi \tag{2.4}$$

2°. Solution of the Laplace's equation for the quadrant $0 < x,\ y < \infty$, with the boundary conditions $u(0,y)=0,\ u(l-0,y)=\mu u(l+0,y),\ u_x(l-0,y)=\nu u_x(l+0,y)$

$$u(x,0)=f(x) \tag{2.5}$$

has the form

$$u(x,y)=\int_0^\infty A(\lambda)\,\varphi(x,\lambda)\,e^{-\lambda y}d\lambda \tag{2.6}$$

where the quantity $A(\lambda)$ is given by (2.4). In particular, if

$$f(x)=T_0\quad(0<x<l),\qquad f(x)=0\quad(l<x<\infty) \tag{2.7}$$

then

$$A(\lambda)=\frac{2}{\pi}\frac{T_0}{\nu\omega(\lambda)}\,\frac{1-\cos\lambda l}{\lambda} \tag{2.8}$$

and the solution of the problem assumes the form

$$u(x,y)=\frac{2T_0}{\pi\nu}\int_0^\infty\frac{1-\cos\lambda l}{\lambda\omega(\lambda)}\,\varphi(x,\lambda)\,e^{-\lambda y}d\lambda. \tag{2.9}$$

3°. We consider the torsion of a semi-infinite cylinder $(0<r<R,\ 0<z<\infty)$ with one face fixed, consisting of two different materials separated by the section $z=l$. Solving the equation

$$\Delta u - r^{-2}\ u=0 \tag{2.10}$$

for the only component $u_\varphi(r,z)\equiv u$ of elastic displacement we find

$$u=\int_0^\infty A(\lambda)\frac{I_1(\lambda r)}{I_1(\lambda R)}\,\varphi(z,\lambda)\,d\lambda,\qquad \delta=\frac{G_1}{G_2} \tag{2.11}$$

where $G_{1,2}$ denote the shear moduli.

In the simplest case when $r=R$ and the displacement $u=f(z)$ is given, the value of $A(\lambda)$ can be found from (2.4). The problems of torsion of a two-layer rod with the stresses on its surface given, are solved in a similar manner.

The expansion (1.3) discussed above represents a generalization of the Fourier sine integral to the case of a compound interval. A similar theorem also exists for the Fourier cosine integral when the condition of the second kind $\varphi'(0)=0$ is laid down in the corresponding boundary value problem for $x=0$. An expansion pertaining to a boundary condition of the third kind [2] which can be proved using a method analogous to that given in Sect. 1, represents a further generalization in the same direction.

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